the zeroth solution of system (5.1). The theorem is proved.

Example 4. Let the determine the conditions of asymptotic stability as a whole of the zeroth solution of system

$$\frac{dx_1}{dt} = a_{1111}x_1^3 + a_{1112}x_1^2x_2 + a_{1122}x_1x_2^2 + a_{1222}x_2^3$$
$$\frac{dx_2}{dt} = a_{2111}x_1^3 + a_{2112}x_1^2x_2 + a_{2122}x_1x_2^2 + a_{2222}x_2^3 \tag{5.9}$$

The derivative with respect to time t of the quadratic form with constant real coefficients

$$v(x_1, x_2) = -{}^1/_2\,[(p_{11}x_1 + p_{12}x_2)^2 + (p_{22}x_2)^2], \quad p_{11} \neq 0, \quad p_{22} \neq 0$$

is, by virtue of system (5.9), a fourth-power form of the variables x_1, x_2, i.e.

$$\frac{dv}{dt} = A_{1111}x_1^4 + A_{1112}x_1^3x_2 + A_{1122}x_1^2x_2^2 + A_{1222}x_1x_2^3 + A_{2222}x_2^4 \tag{5.10}$$

$$A_{1111} = -p_{11}^2a_{1111} - p_{11}p_{12}a_{2111} \tag{5.11}$$
$$A_{1112} = -p_{11}^2a_{1112} - p_{11}p_{12}a_{1111} - p_{11}p_{12}a_{2112} - (p_{12}^2 + p_{22}^2)\,a_{2111}$$
$$A_{1122} = -p_{11}^2a_{1122} - p_{11}p_{12}a_{1112} - p_{11}p_{12}a_{2122} - (p_{12}^2 + p_{22}^2)\,a_{2112}$$
$$A_{1222} = -p_{11}^2a_{1222} - p_{11}p_{12}a_{1122} - p_{11}p_{12}a_{2222} - (p_{12}^2 + p_{22}^2)\,a_{2122}$$
$$A_{2222} = -p_{11}p_{12}a_{1222} - (p_{12}^2 + p_{22}^2)\,a_{2222}$$

In accordance with Theorem 4 the sufficient conditions of asymptotic stability as a whole of the zeroth solution of system (5.1) are the existence of real numbers $p_{11} \neq 0,\ p_{12}, p_{22} \neq 0$ and numbers $a_{1122}, a_{11} \neq 0, a_{12}, a_{13}, a_{22} \neq 0, a_{23}, a_{33} \neq 0$, which satisfy (4.3) and (5.11).

By Theorem 5, the sufficient conditions for asymptotic stability as a whole of the zeroth solution of (5.1) is the existence of a real solution $p_{11} \neq 0, p_{12}, p_{22} \neq 0, b_{111} \neq 0, b_{112}, b_{122}, b_{212} \neq 0, b_{222}$, $b_{322} \neq 0$ of the system of algebraic equations (4.7) and (5.11).

Note that the application of the sufficient conditions for a form of even power to be of fixed sign, based on the Sylvester criterion for quadratic forms to be of fixed sign /1/, leads to the derivation of the sufficient conditions for asymptotic stability as a whole of the zeroth solution of system (5.1) in the form of inequalities for the Sylvester determinants /1/.

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ON THE STABILITY OF INVARIANT MANIFOLDS OF MECHANICAL SYSTEMS*

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The stability of degenerate invariant manifolds of steady motions of mechanical systems imbedded in one another /1/ is investigated using Liapunov's second method.

1. Statement of the problem. Problems of the separation and qualitative investigation of invariant manifolds of the steady motions of autonomous differential equations of mechanical systems

$$x_i^{\cdot} = X_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n \tag{1.1}$$

with smooth right sides in $U \subset R^n$, generated by their first inegrals

$$V_0(x_1, \ldots, x_n) = c_0, \; V_1(x_1, \ldots, x_n) = c_1, \; \ldots, \; V_m(x_1, \ldots, x_n) = c_m \tag{1.2}$$

which are also assumed to be autonomous and smooth (even analytic) in the respective region $V \subset U \subset R^n$ are considered.

Let us set up the "complete" integral of system (1.1)

$$K = \lambda_0V_0(x) + \lambda_1V_1(x) + \ldots + \lambda_mV_m(x)$$

It is always possible to assume one of the quantities $\lambda_j = \text{const}$ in K to be unity. Henceforth, we will assume $\lambda_0 = 1$, since in a general consideration it is not necessary to analyse

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the case $\lambda_0 = 0$ specifically. The equations that, together with (1.2), define sets of invariant manifolds of system (1.1) corresponding to the integral K are

$$y_i = \partial K/\partial x_i = f_i (x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = 0, \ i = 1, 2, \ldots, n \tag{1.3}$$

The Jacobi matrix of the transformation from x to y has the form

$$J = \| \partial y_i/\partial x_j \| = \| \partial^2 K/\partial x_i \partial x_j \|$$

Henceforth, we shall consider those sets of invariant manifolds, defined by (1.3) and (1.2), on which

$$\det J = \Phi (x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = 0 \tag{1.4}$$

Definition 1. The invariant manifolds determined by K are called degenerate if on them $\det J = 0$.

Let a certain degenerate set of invariant manifolds of system (1.1) defined for $k < n$ irrespective of x, by the equations from (1.3)

$$y_1 = f_1 (x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_f) = 0, \ldots, \\ y_k = f_k (x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_f) = 0 \tag{1.5}$$

is given, and are parametrized by the quantities $\lambda_1, \ldots, \lambda_f$ remaining in (1.3), after appropriate allowance for condition (1.4). We select a part of coordinates x_i as the variables on manifold (1.5) with fixed $\lambda_1, \ldots, \lambda_f$. Generally, one these sets may be insufficient to define the entire manifold. We shall then use several such sets in corresponding charts /2/. In each chart, for instance, the chart with coordinates $x_1, \ldots, x_{n-k}$, we can determine the vector field generated on the respective manifold, using the input equations (1.1)

$$x_j^{\cdot} = X_j (x_1, \ldots, x_{n-k}, \lambda_1, \ldots, \lambda_f); \ j = 1, 2, \ldots, n-k$$

We can also write the contraction of integrals (1.2) for this field for a given selection of the independent variables

$$V_0 (x_1, \ldots, x_{n-k}, \lambda_1, \ldots, \lambda_f) = c_0, \ldots, V_l (x_1, \ldots, x_{n-k}, \\ \lambda_1, \ldots, \lambda_f) = c_l$$

which, as a rule, is less than $m + 1$.

In other charts we similarly obtain equations of the form

$$x_{i_1}^{\cdot} = X_{i_1} (x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f), \ldots, \\ x_{i_{n-k}}^{\cdot} = X_{i_{n-k}} (x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) \tag{1.6}$$

and the respective integrals

$$V_0 (x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) = c_0, \ldots, V_l (x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) = c_l \tag{1.7}$$

The transformations of the transition from a chart to a chart of corresponding smoothness must, of course, also be determined.

The above discussion enables us to state the problem of finding the sets of invariant manifolds of second-level steady motions generated on the manifold (1.5) in each chart by the integrals (1.7) of Eqs.(1.6), using the same scheme as for Eqs.(1.1) and (1.2). Continuing this process, we obtain invariant manifolds of the third and higher level. This enables the steady motions to be classified by the imbedding and degree of degeneration. This classification is similar to, Thom's classification of mapping singularities /3/.

Let us consider the question of the steady-motion stability of manifolds of various levels, separated by the method indicated above. It is reasonable to obtain here the sufficient conditions for stability on the basis of the theorems in /4, 5/, using the first integrals of the problem as the Liapunov functions.

We shall specify the concepts that will be necessary subsequently. Let the input degenerate set of steady motions in the enveloping space be defined by system (1.5). We introduce deviations from the manifold (1.5), using the variables $y_1, \ldots, y_k$ which will be further called coordinates "normal" to the manifold. Selecting in each chart $x_{i_1}, \ldots, x_{i_{n-k}}, y_1, \ldots, y_k$ as variables, we write the equations of perturbed motion obtained from (1.1) in the form

$$\begin{aligned} y_1^{\cdot} &= Y_1 (y_1, \ldots, y_k, x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ y_k^{\cdot} &= Y_k (y_1, \ldots, y_k, x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) \\ x_{i_1}^{\cdot} &= X_{i_1} (y_1, \ldots, y_k, x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{i_{n-k}}^{\cdot} &= X_{i_{n-k}} (y_1, \ldots, y_k, x_{i_1}, \ldots, x_{i_{n-k}}, \lambda_1, \ldots, \lambda_f) \end{aligned} \tag{1.8}$$

which when $y_1 = \ldots = y_k = 0$ become system (1.6) which determines the motions on manifold (1.5) on the selected chart.

Let the input system (1.1) have as the first integral

$$V(x_1, \ldots, x_n, \lambda) = c$$

which in one of the charts in coordinates $x_{i_1}, \ldots, x_{i_{n-k}}, y_1, \ldots, y_k$ takes the form

$$V(y_1, \ldots, y_k, \lambda_1, \ldots, \lambda_f) = c \tag{1.9}$$

Obviously, in any other chart the form of integral (1.9) does not, generally, change, since passing from one chart to another reduces to replacing the variables $x_{i_1}, \ldots, x_{i_{n-k}}$, which do not occur in (1.9), by the variables $x_{j_1}, \ldots, x_{j_{n-k}}$.

If integral (1.9) is of fixed sign with respect to $y_1, \ldots, y_k$ when $\lambda_1, \ldots, \lambda_f$ from some set, then on the basis of the slightly modified Rumyantsev's theorem /5/, we can conclude that part of our set (1.5) (or the whole set) of manifolds is stable in the sense of the following definition.

Definition 2. A manifold is stable, if for any fairly small $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that only at the initial instant of time $t = t_0$ for any chart

$$\| y(t_0) \| < \delta \tag{1.10}$$

then for all $t > t_0$ and any chart and along any trajectory that satisfies when $t = t_0$ the condition (1.10), $\| y(t) \| < \varepsilon$. Here $\| \cdot \|$ is the norm of $(y_1^2 + y_2^2 + \ldots + y_k^2)^{1/2}$, or any equivalent to it over all coordinates normal to the manifold.

Remark. It is always possible to treat $\| y \|$ as the distance between the manifold (1.5) and a point of phase space. Hence Definition 2 is the condition of stability of the set formulated in coordinate form /6, 7/.

In a number of problems of solid body dynamics the situation arises when the system of equations of the type of (1.8) admits of a general integral of the form

$$V(x_{i_1}, \ldots, x_{i_{n-k}}, y_1, \ldots, y_k, \lambda_1, \ldots, \lambda_f) = c = 1 \tag{1.11}$$

whose constant is fixed by some (let us say, geometrical) oonsiderations. If the integral (1.11) is in the bundle of *K* integrals that generate the set (1.5), then formula (1.11) follows from the totality of steady motions on (1.5), which is a certain submanifold.

It is often convenient to pose the question of the stability of the complete set of manifolds (1.5), having in view that the submanifold (1.11) separated in (1.5) inherits the stability of the enveloping manifold.

Consider one more possible formulation of the stability problem involving the integral (1.11).

Definition 3. The parametrized set of invariant manifolds (1.5) of the totality of systems (1.8) with integral (1.11) admits of a subset with motions arbitrarily close to zero with respect to the variables x_{i_1}, x_{i_d}, if for any arbitrarily small $|x_{i_1}^\circ|, \ldots, |x_{i_d}^\circ|$ there exists $\lambda_1^\circ, \ldots, \lambda_f^\circ$ such that

$$V(\overset{\circ}{x}_{i_1}, \ldots, \overset{\circ}{x}_{i_d}, x_{i_{d+1}}, \ldots, x_{i_{n-k}}, 0, \ldots, 0, \lambda_1^\circ, \ldots, \lambda_f^\circ) = 1$$

Definition 4. The parametrized set of manifolds (1.5) of the totality of systems (1.8) with the integral (1.11) admits a stable subset of manifolds with motions that remain arbitrarily close to zero with respect to the variables $x_{i_1}, \ldots, x_{i_d}$, if the following conditions are satisfied.

1°. Systems (1.8) with the integral (1.11) admit of the subset of a family of manifolds (1.5) with properties outlined in Definition 3.

2°. For any $\varepsilon > 0$ we can find $\delta_1(\varepsilon) > 0$, $\delta_2(\varepsilon) > 0$ such that if

$$\sum_{j=1}^{d} x_{i_j}^2(t_0) < \delta_1, \quad \sum_{i=1}^{k} y_i^2(t_0) < \delta_2$$

then

$$\sum_{j=1}^{d} x_{i_j}^2(t) + \sum_{i=1}^{k} y_i^2(t) < \varepsilon$$

with an appropriate selection of $\lambda_1^\circ, \ldots, \lambda_f^\circ$ from some set for all $t > t_0$.

To obtain the sufficient conditions of existence in the system of a subset of invariant manifolds with properties indicated in Definition 4, the theorems, which are a slight modification of theorems in /5/ on the stability with respect to a part of the variables, can be used. The following statement will be sufficient subsequently. If Eqs.(1.8) of perturbed motion with integral (1.11) admit of in the respective charts the integral

$$W(x_{i_1}, \ldots, x_{i_d}, y_1, \ldots, y_k, \lambda_1, \ldots, \lambda_f)$$

which is of fixed sign with respect to the variables $y_1, \ldots, y_k, x_{i_1}, \ldots, x_{i_d}$ for values $\lambda_1, \ldots, \lambda_f$ from the set, where the properties postulated in Definition 3 are realized, the system has a subset of manifolds with properties postulated by Definition 4.

The proof of this statement follows the standard scheme of proof of theorems on stability in Liapunov's second method.

2. Invariant manifolds of the Kovalevskaya top. Let us consider some examples of stability investigations of degenerate invariant manifolds of steady motions in the dynamics of a solid. Let the mass distribution of the body with a fixed point satisfy the Kovalevskaya conditions ($A = B = 2C$, $x_0 \neq 0$, $y_0 = z_0 = 0$). The Kovalevskaya integral /8/

$$V_2 = (p^2 - q^2 - x_0\gamma_1)^2 + (2pq - x_0\gamma_2)^2 = k^2$$

generates in this case an invariant manifold of the Delaunay steady motions which is defined in the phase space of the system by two equations

$$y_1 = p^2 - q^2 - x_0\gamma_1 = 0, \quad y_2 = 2pq - x_0\gamma_2 = 0 \tag{2.1}$$

and has, therefore, a degree of degeneration of four. The initial Euler-Poisson differential equations in this problem for such manifold give the following equations of perturbed motion:

$$\begin{gathered} 2p^{\cdot} = qr, \ 2q^{\cdot} = -rp + x_0\gamma_3, \ r^{\cdot} = y_2 - 2pq \\ \gamma_3^{\cdot} = -[q(p^2 + q^2) + qy_1 - py_2]/x_0, \ y_1^{\cdot} = ry_2, \ y_2^{\cdot} = -ry_1 \end{gathered} \tag{2.2}$$

which, when $y_1 = y_2 = 0$, define a vector field on the manifold (2.1) itself

$$2p^{\cdot} = qr, \quad 2q^{\cdot} = -rp + x_0\gamma_3, \quad r^{\cdot} = -2pq, \quad \gamma_3^{\cdot} = -q(p^2 + q^2)/x_0 \tag{2.3}$$

The latter admit of the integrals

$$\begin{gathered} 2H = 4p^2 + r^2 = 2h, \ V_1 = r\gamma_3 + 2p(p^2 + q^2)/x_0 = m \\ V_3 = \gamma_3^2 + (p^2 + q^2)/x_0^2 = 1 \end{gathered} \tag{2.4}$$

which are a contraction onto the set of Delaunay motions of the classic integrals of the Kovalevskaya problem /9/. The total integral

$$K = H - \nu_1 V_1 - {}^1/_2\nu_2 V_3$$

of Eqs.(23) formed from expressions (2.4), generates invariant manifolds of steady motions of the second level that lie on the set (2.1).

Among the second level manifolds generated by K, we note the following: "pendulum oscillations" around the horizontal y axis

$$4p = 0, \ r = 0, \ \nu_1 = \nu_2 = 0 \tag{2.5}$$

that are defined by the differential equations

$$2q^{\cdot} = x_0\gamma_3, \quad \gamma_3^{\cdot} = -q^3/x_0$$

and the motions which lie on the one-parameter set defined by the equations

$$2\nu_1 x_0 p - \nu_1^2(p^2 + q^2) = 0, \quad r - \nu_1\gamma_3 = 0, \ \nu_1^2 + \nu_2 = 0 \tag{2.6}$$

Since a unique system of coordinates cannot be introduced on this cylindrical surface, it is reasonable to use several such sets when $\nu_1 \neq 0$ and $\nu_1 x_0 > 0$

$$\begin{gathered} 1°.\ r, p;\ 0 < p < 2x_0/\nu_1,\ 0 < q \leqslant x_0/\nu_1,\ q = P \\ 2°.\ r, p;\ 0 < p < 2x_0/\nu_1,\ -x_0/\nu_1 \leqslant q < 0,\ q = -P \\ 3°.\ r, q;\ -x_0/\nu_1 < q < x_0/\nu_1,\ x_0/\nu_1 < p \leqslant 2x_0/\nu_1,\ p = \frac{x_0}{\nu_1} + Q \\ 4°.\ r, q;\ -x_0/\nu_1 < q < x_0/\nu_1,\ 0 \leqslant p < x_0/\nu_1,\ p = \frac{x_0}{\nu_1} - Q \\ \left(P = \left(\frac{2px_0}{\nu_1} - p^2\right)^{1/2},\ Q = \left(\frac{x_0^2}{\nu_1^2} - q^2\right)^{1/2}\right) \end{gathered} \tag{2.7}$$

The situtation in the case of $\nu_1 x_0 < 0$ is exactly the same.

In each of the charts (2.7) on manifolds (2.6) we use Eqs.(2.3) to define the vector field

$$\begin{gathered} 1°.\ 2p^{\cdot} = rP, \quad r^{\cdot} = -2pP \\ 2°.\ 2p^{\cdot} = -rP, \quad r^{\cdot} = 2pP \\ 3°.\ 2q^{\cdot} = -rQ,\ r^{\cdot} = -2q\left(\frac{x_0}{\nu_1} + Q\right) \\ 4°.\ 2q^{\cdot} = rQ,\ r^{\cdot} = -2q\left(\frac{x_0}{\nu_1} - Q\right) \end{gathered} \tag{2.8}$$

For the first two charts the vector field admits of the first integral

$$4p^2 + r^2 = \nu_1^2 \tag{2.9}$$

which in the third and fourth charts takes the form

$$4\left(\frac{x_0}{\nu_1}+Q\right)^2+r^2=\nu_1^2 \tag{2.10}$$

$$4\left(\frac{x_0}{\nu_1}-Q\right)^2+r^2=\nu_1^2 \tag{2.11}$$

respectively. The integral (2.9), (2.10), and (2.11) are of type (1.11) for each fixed ν_1^2. It may also be used to determine the manifolds of the third level steady motions that lie on (2.6).

In the third chart the integral (2.10) generates as the third-level manifold, the permanent rotation

$$r=q=0\left(p=\frac{2x_0}{\nu_1}=\sqrt{x_0},\ \nu_1=2\sqrt{x_0}\right) \tag{2.12}$$

where the parameter ν_1 is determined after substituting the solution into integral (2.10).

In the fourth chart the third level solution could have been

$$r=q=0 \quad (p=0)$$

but it does not satisfy integral (2.10) when substituted into it, when $\nu_1 \neq 0$ and, consequently, is not a steady motion.

We revert now to the analysis of the system of equations of invariant manifolds that are cut out by integrals (2.9), (2.10), and (2.11) on the surface (2.6) $(\nu_1 x_0 > 0)$. It can be seen that for each fixed $\nu_1 > 2\sqrt{x_0}$ the intersection of the two cylinders (2.6) and (2.9) defines two trajectories that envelop the first of them; $\nu_1 = 2\sqrt{x_0}$ corresponds to two separatrix curves that adjoin the permanent rotation (2.12) at both ends; finally, when $0 < \nu_1 < 2\sqrt{x_0}$ we obtain closed curves that lie on surface (2.6) and contain the point $p=q=r=0$. When $\nu_1 = 0$ the manifold (2.6) becomes degenerate and we obtain here, as the limit of an invariant manifold, a set of pendulum oscillations (2.5).

Questions of singularities of families of sets of steady motions with respect to the parameters are not considered here.

3. Stability. Let us now investigate the sets of steady motions described above. Equations (2.2) admit of the first integral /8/

$$\Delta V_2 = y_1^2 + y_2^2 \gg 0$$

and, consequently, the whole manifold of the Delaunay steady motions is stable in the system phase space.

We shall consider the question of the stability of the parametrized system of manifolds defined by Eqs.(2.6) relative to the Delaunay motions. Introducing the "normal coordinates"

$$z_1 = 2px_0 - \nu_1(p^2+q^2), \quad z_2 = r - \nu_1\gamma_3$$

we write the perturbed motion of this set relative to the manifold (2.1), using all four charts on (2.6). Thus, the necessary equations in the fourth chart have the form

$$z_1^{\cdot} = x_0 q z_2, \quad r^{\cdot} = -2q\left(\frac{x_0}{\nu_1}-R\right) \tag{3.1}$$

$$z_2^{\cdot} = -qz_1/x_0, \quad 2q^{\cdot} = -\frac{x_0 z_2}{\nu_1} + rR$$

$$R = [(x_0/\nu_1)^2 - q^2 - (z_1/\nu_1)]^{1/2}$$

Similar equations also occur in the three remaining charts.

A simple check will show that Eqs.(3.1) admit of the integral

$$2\Delta K = z_2^2 + z_1^2/x_0^2$$

which obviously occurs in the remaining charts also. It enables us to conclude that the manifold (2.6) is stable relative to the Delaunay motions.

Since the real set (2.6), (2.9) differs from (2.6) only by the setting of the constant of the integral of cosines, this parametrized manifold is also stable with respect to the variables z_1 and z_2 relative to the Delaunay motions for any positive parameter $\nu_1 \neq 0$.

Note that the equations of perturbed motion in the fourth chart (3.1) admit of the following two integrals:

$$\Delta V_3 = \frac{(2px_0 - z_1)^2}{x_0^2\nu_1^2} + \frac{r^2}{\nu_1^2} = 1 \tag{3.2}$$

$$\Delta W = z_2^2 + rz_2 + \frac{r^2}{2} + \frac{z_1^2}{2x_0^2} + 4p^2 \tag{3.3}$$

It follows directly from (3.2) that for $z_1 = z_2 = 0$ and any arbitrarily small $|p^\circ|$ and $|r^\circ|$, it is always possible by a proper selection of ν_1° to satisfy the relation

$$4p^{\circ 2}/\nu_1^{\circ 2} + r^{\circ 2}/\nu_1^{\circ 2} = 1 \ \ (p = (x_0/\nu_1) - Q)$$

i.e. here the conditions of Definition 3 for small $|p^\circ|$ and $|r^\circ|$ are satisfied on manifold (2.6).

The integral (3.3) is of fixed sign with respect to the variables z_1, z_2, p, r for any $\nu_1 > 0$. We may, thus, speak of the stability of part of the set of manifolds (2.6) and (2.11) with small motions with respect p and r for a proper selection of the parameter ν_1° in the sense of Definition 4.

Let us also consider the stability of the invariant manifold of pendulum oscillations of the body (2.5) relative to the Delaunay manifold. Here the equations of perturbed motion are the same as the equations of motion on the Delaunay manifold (2.3). The variables p and r here play the part of coordinates normal to the manifold (2.5). When $p = r = 0$, we obtain from Eqs.(2.3) the vector field on the set of pendulum oscillations (2.5).

Equations (2.3) admit of the first integral

$$H = 2p^2 + r^2/2$$

which is of fixed sign with respect to normal coordinates. The stability of the set of motions (2.5) relative to the Delaunay manifold follows from this formula. Note that the limit manifold (2.5) for the stable set (2.6) is also stable.

Finally, let us consider the question of when it is possible to conclude from the stability of a lower level manifold and the stability in it of a higher level manifold, that the latter is stable in the initial phase space.

Definition 5. A second level manifold is stable in the phase space of the system, if it is stable with respect to normal coordinates of the first and second levels.

Let us restrict the problem to the specific example of obtaining the sufficient conditions of such stability, using Lyapunov's function in the form of a bundle of first integrals.

Consider the stability of pendulum oscillations (2.5) in the phase space of a solid. We write the equations of perturbed motion of the problem, using the normal coordinates of manifold (2.5) in the Delaunay manifold (p, r) and the normal coordinates (y_1, y_2) of the Delaunay manifold in phase space. The necessary equations are obtained from the Euler-Poisson equations of the initial Kovalevskaya problem in the form

$$y_1^{\cdot} = ry_2, \quad y_2^{\cdot} = -ry_1, \; 2p^{\cdot} = qr, \quad 2q^{\cdot} = -rp + x_0\gamma_3 \tag{3.4}$$
$$r^{\cdot} = -2pq + y_2, \; \gamma_3^{\cdot} = -[q(p^2+q^2) + qy_1 - py_2]/x_0$$

Here, the unperturbed solution is $p = r = y_1 = y_2 = 0$, with the vector field in it

$$2q^{\cdot} = x_0\gamma_3, \quad \gamma_3 = -q^3/x_0$$

As was shown above, Eqs.(3.4) admit of the integrals

$$\Delta V_2 = y_1^2 + y_2^2, \quad 2\Delta H = 4p^2 + r^2 - 2y_1 \tag{3.5}$$

The last integral is the perturbation of the first integral (2.4), when the Delaunay manifold is varied.

It is seen that from the integrals (3.5) it is possible to form the bundle

$$L = \Delta H^2 - \varkappa \Delta V_2 = {}^1/_4\,(4p^2 + r^2 - 2y_1)^2 - \varkappa\,(y_1^2 + y_2^2)$$

which, when $\varkappa < -1$, is a positive-definite function with respect to all normal coordinates of (3.4).

It is, thus, possible to draw conclusions regarding the stability of pendulum oscillations of a body (2.5) in phase space, using the slightly modified Rumyantsev theorem.

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